1 Limits

WHAT IS A LIMIT?

In order to understand calculus, you need to know what a "limit" is. A limit is the value a function (which usually is written "$f(x)$" on the AP exam) approaches as the variable within that function (usually "x") gets nearer and nearer to a particular value. In other words, when x is very close to a certain number, what is $f(x)$ very close to? As far as the AB test is concerned, that's all you have to know about evaluating limits. There's a more technical method that you BC students have to learn, but we won't discuss it until we have to—at the end of this chapter.

Let's look at an example of a limit: What is the limit of the function $f(x) = x^2$ as x approaches 2?

In limit notation, the expression "the limit of $f(x)$ as x approaches 2" is written like this: $\lim_{x \to 2} f(x)$. In order to evaluate the limit, let's check out some values of $\lim_{x \to 2} f(x)$ as x increases and gets closer to 2 (without ever exactly getting there).

When $x = 1.9$, $f(x) = 3.61$.

When $x = 1.99$, $f(x) = 3.9601$.

When $x = 1.999$, $f(x) = 3.996001$.

When $x = 1.9999$, $f(x) = 3.99960001$.

As x increases and approaches 2, $f(x)$ gets closer and closer to 4. This is called the **left-hand limit** and is written: $\lim_{x \to 2^-} f(x)$. Notice the little minus sign!

What about when x is bigger than 2?

When $x = 2.1$, $f(x) = 4.41$.

When $x = 2.01$, $f(x) = 4.0401$.

When $x = 2.001$, $f(x) = 4.004001$.

When $x = 2.0001$, $f(x) = 4.00040001$.

As x decreases and approaches 2, $f(x)$ still approaches 4. This is called the **right-hand limit** and is written like this: $\lim_{x \to 2^+} f(x)$. Notice the little plus sign!

We got the same answer when evaluating both the left- and right-hand limits, because when x is 2, $f(x)$ is 4. You should always check both sides of the independent variable because, as you'll see shortly, sometimes you don't get the same answer. Therefore, we write that $\lim_{x \to 2} x^2 = 4$.

We didn't really need to look at all of these decimal values to know what was going to happen when x got really close to 2. But it's important to go through the exercise because, typically, the answers get a lot more complicated. Let's do a few examples.

Example 1: Find $\lim_{x \to 5} x^2$.

The approach is simple: plug in 5 for x, and you get 25.

Example 2: Find $\lim_{x \to 3} x^3$.

Here the answer is 27.

There are some simple algebraic rules of limits that you should know. These are:

$$\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$$

Example: $\lim_{x \to 5} 3x^2 = 3 \lim_{x \to 5} x^2 = 75$

from the left and the right. But when we have $\frac{k}{x}$, the function's sign depends on the sign of x, and you get a different limit from each side.

Let's look at a few examples in which the independent variable approaches infinity.

Example 6: Find $\lim_{x\to\infty}\frac{1}{x}$.

As x gets bigger and bigger, the value of the function gets smaller and smaller. Therefore, $\lim_{x\to\infty}\frac{1}{x}=0$.

Example 7: Find $\lim_{x\to-\infty}\frac{1}{x}$.

It's the same situation as the one in Example 6; as x decreases (approaches negative infinity), the value of the function increases (approaches zero). We write this:

$$\lim_{x\to-\infty}\frac{1}{x}=0$$

We don't have the same problem here that we did when x approached zero because "positive zero" is the same thing as "negative zero," whereas positive infinity is different from negative infinity.

Here's another rule:

> If k and n are constants, $|x|>1$, and $n>0$, then $\lim_{x\to\infty}\frac{k}{x^n}=0$, and $\lim_{x\to-\infty}\frac{k}{x^n}=0$.

Example 8: Find $\lim_{x\to\infty}\frac{3x+5}{7x-2}$.

When you have variables in both the top and the bottom, you can't just plug ∞ into the expression. You'll get $\frac{\infty}{\infty}$. We solve this by using the following technique:

When an expression consists of a polynomial divided by another polynomial, divide each term of the numerator and the denominator by the highest power of x that appears in the expression.

The highest power of x in this case is x^1, so we divide every term in the expression (both top and bottom) by x, like so:

$$\lim_{x\to\infty}\frac{3x+5}{7x-2}=\lim_{x\to\infty}\frac{\frac{3x}{x}+\frac{5}{x}}{\frac{7x}{x}-\frac{2}{x}}=\lim_{x\to\infty}\frac{3+\frac{5}{x}}{7-\frac{2}{x}}$$

Now when we take the limit, the two terms containing x approach zero. We're left with $\frac{3}{7}$.

Example 9: Find $\lim_{x\to\infty}\frac{8x^2-4x+1}{16x^2+7x-2}$

Divide each term by x^2. You get:

$$\lim_{x\to\infty}\frac{8-\frac{4}{x}+\frac{1}{x^2}}{16+\frac{7}{x}-\frac{2}{x^2}}=\frac{8}{16}=\frac{1}{2}$$

Example 10: Find $\lim_{x\to\infty}\frac{-3x^{10}-70x^5+x^3}{33x^{10}+200x^8-1000x^4}$.

Here, divide each term by x^{10}:

$$\lim_{x\to\infty}\frac{-3x^{10}-70x^5+x^3}{33x^{10}+200x^8-1000x^4}=\lim_{x\to\infty}\frac{-3-\frac{70}{x^5}+\frac{1}{x^7}}{33+\frac{200}{x^2}-\frac{1000}{x^6}}=-\frac{3}{33}=-\frac{1}{11}$$

Focus your attention on the highest power of x. The other powers don't matter, because they're all going to disappear. Now we have three new rules for evaluating the limit of a rational expression as x approaches infinity:

(1) If the highest power of x in a rational expression is in the numerator, then the limit as x approaches infinity is infinity.

Example: $\lim_{x\to\infty}\frac{5x^7-3x}{16x^6-3x^2}=\infty$

(2) If the highest power of x in a rational expression is in the denominator, then the limit as x approaches infinity is zero.

Example: $\lim_{x\to\infty}\frac{5x^6-3x}{16x^7-3x^2}=0$

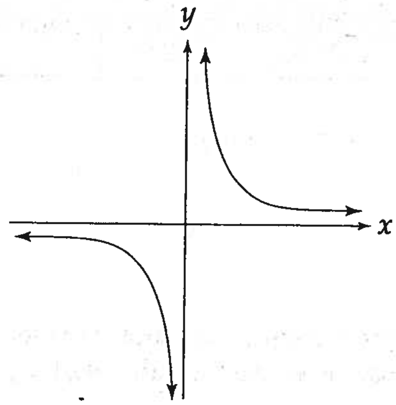
(3) If the highest power of x in a rational expression is the same in both the numerator and denominator, then the limit as x approaches infinity is the coefficient of the highest term in the numerator divided by the coefficient of the highest term in the denominator.

Example: $\lim_{x\to\infty}\frac{5x^7-3x}{16x^7-3x^2}=\frac{5}{16}$

Example 5: Find $\lim_{x \to 0} \frac{1}{x}$.

Here you have a problem. If you plug in some very small positive values for x (0.1, 0.01, 0.001, etc.), you approach ∞. In other words, $\lim_{x \to 0^+} \frac{1}{x} = \infty$. But, if you plug in some very small negative values for x (−0.1, −0.01, −0.001, etc.) you approach $-\infty$. That is, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. Because the right-hand limit is not equal to the left-hand limit, this limit does not exist.

Look at the graph of $\frac{1}{x}$:



You can see that on the left side of $x = 0$, the curve approaches $-\infty$, and on the right side of $x = 0$, the curve approaches ∞. There are some very important points that we need to emphasize from the last two examples.

(1) If the left-hand limit of a function is not equal to the right-hand limit of the function, then the limit does not exist.

(2) A limit equal to infinity is not the same as a limit that does not exist, but sometimes you will see the expression "no limit," which serves both purposes. If $\lim_{x \to a} f(x) = \infty$, the limit, technically, does not exist.

(3) If k is a positive constant, then $\lim_{x \to 0^+} \frac{k}{x} = \infty$, $\lim_{x \to 0^-} \frac{k}{x} = -\infty$, and $\lim_{x \to 0} \frac{k}{x}$ does not exist.

(4) If k is a positive constant, then $\lim_{x \to 0^+} \frac{k}{x^2} = \infty$, $\lim_{x \to 0^-} \frac{k}{x^2} = \infty$, and $\lim_{x \to 0} \frac{k}{x^2} = \infty$.

Why do we state the limit in Example 4 but not for Example 5? Because when we have $\frac{k}{x^2}$, the function is always positive no matter what the sign of x is and thus the function has the same limit

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then

$$\lim_{x\to a}\left[f(x)+g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Example: $\lim_{x\to 5}\left[x^2 + x^3\right] = \lim_{x\to 5} x^2 + \lim_{x\to 5} x^3 = 150$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then

$$\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

Example: $\lim_{x\to 5}\left[\left(x^2+1\right)\sqrt{x-1}\right] = \lim_{x\to 5}\left(x^2+1\right)\lim_{x\to 5}\sqrt{x-1} = 52$

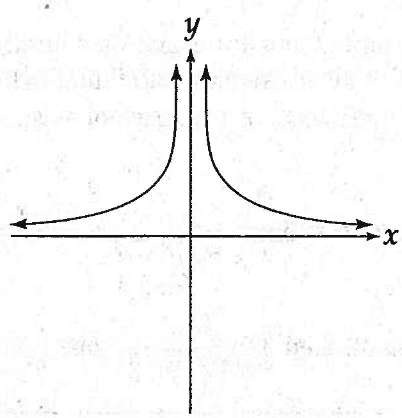
Example 3: Find $\lim_{x\to 0}\left(x^2+5x\right)$.

Plug in 0, and you get 0.

So far, so good. All you do to find the limit of a simple polynomial is plug in the number that the variable is approaching and see what the answer is. Naturally, the process can get messier—especially if x approaches zero.

Example 4: Find $\lim_{x\to 0}\frac{1}{x^2}$.

If you plug in some very small values for x, you'll see that this function approaches ∞. And it doesn't matter whether x is positive or negative, you still get ∞. Look at the graph of $y = \frac{1}{x^2}$:



On either side of $x = 0$ (the y-axis), the curve approaches ∞.

LIMITS OF TRIGONOMETRIC FUNCTIONS

At some point during the exam, you'll have to find the limit of certain trig expressions, usually as x approaches either zero or infinity. There are four standard limits that you should memorize—with those, you can evaluate all of the trigonometric limits that appear on the test. As you'll see throughout this book, calculus requires that you remember all of your trig from previous years.

$$\text{Rule No. 1: } \lim_{x\to 0}\frac{\sin x}{x}=1 \text{ (}x\text{ is in radians, \textit{not} degrees)}$$

This may seem strange, but if you look at the graphs of $f(x) = \sin x$ and $f(x) = x$, they have approximately the same slope near the origin (as x gets closer to zero). Since x and the sine of x are about the same as x approaches zero, their quotient will be very close to one. Furthermore, because $\lim_{x\to 0}\cos x=1$ (review cosine values if you don't get this!), we know that $\lim_{x\to 0}\tan x=\lim_{x\to 0}\frac{\sin x}{\cos x}=0$.

Now we will find a second rule. Let's evaluate the limit $\lim_{x\to 0}\frac{\cos x-1}{x}$. First, multiply the top and bottom by $\cos x+1$. We get: $\lim_{x\to 0}\frac{\cos x-1}{x}\frac{\cos x+1}{\cos x+1}$. Now simplify the limit to: $\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}$. Next, we can use the trigonometric identity $\sin^2 x=1-\cos^2 x$ and rewrite the limit as: $\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}$. Now, break this into two limits: $\lim_{x\to 0}\frac{-\sin x}{x}\frac{\sin x}{(\cos x+1)}$. The first limit is -1 (see Rule No. 1) and the second is 0 (why?), so the limit is 0.

$$\text{Rule No. 2: } \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

Example 11: Find $\lim_{x\to 0}\frac{\sin 3x}{x}$.

Use a simple trick: Multiply the top and bottom of the expression by 3. This gives us: $\lim_{x\to 0}\frac{3\sin 3x}{3x}$. Next, substitute a letter for $3x$; for example, a. Now, we get the following:

$$\lim_{a\to 0}\frac{3\sin a}{a}=3\lim_{a\to 0}\frac{\sin a}{a}=3(1)=3$$

Example 12: Find $\lim_{x\to 0}\frac{\sin 5x}{\sin 4x}$.

Now we get a bit more sophisticated. First, divide both the numerator and the denominator by x, like so:

$$\lim_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\sin 4x}{x}}$$

Next, multiply the top and bottom of the numerator by 5, and the top and bottom of the denominator by 4, which gives us:

$$\lim_{x\to 0}\frac{\frac{5\sin 5x}{5x}}{\frac{4\sin 4x}{4x}}$$

From the work we did in Example 11, we can see that this limit is $\frac{5}{4}$.
Guess what! You have two more rules!

> Rule No. 3: $\lim_{x\to 0}\frac{\sin ax}{x}=a$
>
> Rule No. 4: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\frac{a}{b}$

Example 13: Find $\lim_{x\to 0}\frac{x^2}{1-\cos^2 x}$.

Using trigonometric identities, you can replace $(1 - \cos^2 x)$ with $\sin^2 x$:

$$\lim_{x\to 0}\frac{x^2}{1-\cos^2 x}=\lim_{x\to 0}\frac{x^2}{\sin^2 x}=\lim_{x\to 0}\left(\frac{x}{\sin x}\cdot\frac{x}{\sin x}\right)=1\cdot 1=1$$

Here are other examples for you to try, with answers right beneath them. Give 'em a try, and check your work.

Problem 1. Find $\lim_{x\to 3}\frac{x-3}{x+2}$.

Answer: If you plug in 3 for x, you get $\lim_{x\to 3}\frac{3-3}{3+2}=\frac{0}{5}=0$.

Problem 2. Find $\lim_{x\to 3}\frac{x+2}{x-3}$.

Answer: The left-hand limit is: $\lim_{x\to 3^-}\frac{x+2}{x-3}=-\infty$

The right-hand limit is: $\lim_{x\to 3^+}\frac{x+2}{x-3}=\infty$

These two limits are not the same. Therefore, the limit does not exist.

Problem 3. Find $\lim_{x\to 3}\frac{x+2}{(x-3)^2}$.

Answer: The left-hand limit is: $\lim_{x\to 3^-}\frac{x+2}{(x-3)^2}=\infty$

The right-hand limit is: $\lim_{x\to 3^+}\frac{x+2}{(x-3)^2}=\infty$

These two limits are the same, so the limit is ∞.

Problem 4. Find $\lim_{x\to -4}\frac{x^2+6x+8}{x+4}$.

Answer: If you plug –4 into the top and bottom, you get $\frac{0}{0}$. You have to factor the top into $(x+2)(x+4)$ to get this: $\lim_{x\to -4}\frac{(x+2)(x+4)}{(x+4)}$

Now it's time to cancel like terms: $\lim_{x\to -4}\frac{(x+2)(x+4)}{(x+4)}=\lim_{x\to -4}(x+2)=-2$

Problem 5. Find $\lim_{x\to\infty}\frac{15x^2-11x}{22x^2+4x}$.

Answer: Divide each term by x^2: $\lim_{x\to\infty}\frac{15x^2-11x}{22x^2+4x}=\lim_{x\to\infty}\frac{15-\frac{11}{x}}{22+\frac{4}{x}}=\frac{15}{22}$

Problem 6. Find $\lim_{x\to 0}\frac{4x}{\tan x}$.

Answer: Replace $\tan x$ with $\frac{\sin x}{\cos x}$, which changes the expression into:

$$\lim_{x\to 0}\frac{4x}{\tan x}=\lim_{x\to 0}\frac{4x}{\frac{\sin x}{\cos x}}=\lim_{x\to 0}\frac{4x\cos x}{\sin x}$$

Since $\lim_{x\to 0}\frac{x}{\sin x}=1$ and $\lim_{x\to 0}\cos x=1$, the answer is 4.

Note: Pay careful attention to this next solved problem. It will be very important when you work on problems in Chapter 4.

Problem 7. Find $\lim_{h\to 0}\frac{(5+h)^2-25}{h}$.

Answer: First, expand and simplify the numerator like this:

$$\lim_{h\to 0}\frac{(5+h)^2-25}{h}=\lim_{h\to 0}\frac{25+10h+h^2-25}{h}=\lim_{h\to 0}\frac{10h+h^2}{h}.$$

Next, factor h out of the numerator and the denominator like this:

$$\lim_{h\to 0}\frac{10h+h^2}{h}=\lim_{h\to 0}\frac{h(10+h)}{h}=\lim_{h\to 0}(10+h).$$

Taking the limit you get: $\lim_{h\to 0}(10+h)=10$.

PRACTICE PROBLEM SET 1

Try these 30 problems to test your skill with limits. The answers are in Chapter 21.

1. $\lim_{x\to 8}\left(x^2-5x-11\right)=$

2. $\lim_{x\to 5}\left(\frac{x+3}{x^2-15}\right)=$

3. $\lim_{x\to 0}\pi^2=$

4. $\lim_{x\to 3}\left(\frac{x^2-2x-3}{x-3}\right)=$

5. $\lim_{x\to\infty}\left(\frac{10x^2+25x+1}{x^4-8}\right)=$

6. $\lim_{x\to\infty}\left(\frac{x^4-8}{10x^2+25x+1}\right)=$

7. $\lim_{x\to\infty}\left(\frac{x^4-8}{10x^4+25x+1}\right)=$

8. $\lim_{x\to\infty}\left(\frac{\sqrt{5x^4+2x}}{x^2}\right)=$

9. $\lim_{x\to6^+}\left(\frac{x+2}{x^2-4x-12}\right)=$

10. $\lim_{x\to6^-}\left(\frac{x+2}{x^2-4x-12}\right)=$

11. $\lim_{x\to6}\left(\frac{x+2}{x^2-4x-12}\right)=$

12. $\lim_{x\to0^+}\left(\frac{x}{|x|}\right)=$

13. $\lim_{x\to0^-}\left(\frac{x}{|x|}\right)=$

14. $\lim_{x\to7^+}\left(\frac{x}{x^2-49}\right)=$

15. $\lim_{x\to7}\left(\frac{x}{x^2-49}\right)=$

16. $\lim_{x\to7}\frac{x}{(x-7)^2}=$

17. Let $f(x)=\begin{cases}x^2-5, x\le 3\\ x+2, x>3\end{cases}$

Find: (a) $\lim_{x\to3^-} f(x)$; (b) $\lim_{x\to3^+} f(x)$; and (c) $\lim_{x\to3} f(x)$

18. Let $f(x)=\begin{cases}x^2-5, x\le 3\\ x+1, x>3\end{cases}$

Find: (a) $\lim_{x\to3^-} f(x)$; (b) $\lim_{x\to3^+} f(x)$; and (c) $\lim_{x\to3} f(x)$

19. Find $\lim_{x \to \frac{\pi}{4}} 3\cos x$.

20. Find $\lim_{x \to 0} 3\frac{x}{\cos x}$.

21. Find $\lim_{x \to 0} 3\frac{x}{\sin x}$.

22. Find $\lim_{x \to 0} \frac{\sin 3x}{\sin 8x}$.

23. Find $\lim_{x \to 0} \frac{\tan 7x}{\sin 5x}$.

24. Find $\lim_{x \to \infty} \sin x$.

25. Find $\lim_{x \to \infty} \sin \frac{1}{x}$.

26. Find $\lim_{x \to 0} \frac{x^2 \sin x}{1 - \cos^2 x}$.

27. Find $\lim_{x \to 0} \frac{\sin^2 7x}{\sin^2 11x}$.

28. Find $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$.

29. Find $\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$.

30. Find $\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$.

2

Continuity

Every AP exam has a few questions on continuity, so it's important to understand the basic idea of what it means for a function to be continuous. The concept is very simple: If the graph of the function doesn't have any breaks or holes in it within a certain interval, the function is continuous over that interval.

Simple polynomials are continuous everywhere; it's the other ones—trigonometric, rational, piecewise—that might have continuity problems. Most of the test questions concern these last types of functions. In order to learn how to test whether a function is continuous, you'll need some more mathematical terminology.

THE DEFINITION OF CONTINUITY

In order for a function $f(x)$ to be continuous at a point $x = c$, it must fulfill *all three* of the following conditions:

Condition 1: $f(c)$ exists.

Condition 2: $\lim\limits_{x \to c} f(x)$ exists.

Condition 3: $\lim\limits_{x \to c} f(x) = f(c)$

Let's look at a simple example of a continuous function. (Incidentally, you'll find that these functions are continuous almost everywhere, and the only possible difficulty will occur at a few specific values of x.)

Example 1: Is the function $f(x) = \begin{cases} x+1, & x < 2 \\ 2x-1, & x \geq 2 \end{cases}$ continuous at the point $x = 2$?

Condition 1: Does $f(2)$ exist?
Yes. It's equal to $2(2) - 1 = 3$.

Condition 2: Does $\lim\limits_{x \to 2} f(x)$ exist?

You need to look at the limit from both sides of 2. The left-hand limit is: $\lim\limits_{x \to 2^-} f(x) = 2 + 1 = 3$. The right-hand limit is: $\lim\limits_{x \to 2^-} f(x) = 2(2) - 1 = 3$.

Because the two limits are the same, the limit exists.

Condition 3: Does $\begin{cases} x+1, & x < 2 \\ 2x - 1, & x > 2 \end{cases} f(x) = f(2)$?

The two equal each other, so yes; the function is continuous at $x = 2$.

A simple and important way to check whether a function is continuous is to sketch the function. If you can't sketch the function without lifting your pencil from the paper at some point, then the function is not continuous.

Now let's look at some examples of functions that are not continuous.

Example 2: Is the function $f(x) = \begin{cases} x+1, & x < 2 \\ 2x - 1, & x > 2 \end{cases}$ continuous at $x = 2$?

Condition 1: Does $f(2)$ exist?

Nope. The function of x is defined if x is greater than or less than 2, but not if x is equal to 2. Therefore, the function is not continuous at $x = 2$. Notice that we don't have to bother with the other two conditions. Once you find a problem, the function is automatically not continuous, and you can stop.

Example 3: Is the function $f(x) = \begin{cases} x+1, & x<2 \\ 2x+1, & x \geq 2 \end{cases}$ continuous at $x = 2$?

Condition 1: Does $f(x)$ exist?
Yes. It is equal to 2(2) + 1 = 5.

Condition 2: Does $\lim_{x\to 2} f(x)$ exist?

The left-hand limit is: $\lim_{x\to 2^-} f(x) = 2 + 1 = 3$.

The right-hand limit is: $\lim_{x\to 2^+} f(x) = 2(2) + 1 = 5$.

The two limits don't match, so the limit doesn't exist and the function is not continuous at $x = 2$.

Example 4: Is the function $f(x) = \begin{cases} x+1, & x<2 \\ x^2, & x=2 \\ 2x-1, & x>2 \end{cases}$ continuous at $x = 2$?

Condition 1: Does $f(2)$ exist?
Yes. It's equal to $2^2 = 4$.

Condition 2: Does $\lim_{x\to 2} f(x)$ exist?

The left-hand limit is: $\lim_{x\to 2^-} f(x) = 2 + 1 = 3$.

The right-hand limit is: $\lim_{x\to 2^+} f(x) = 2(2) - 1 = 3$.

Because the two limits are the same, the limit exists.

Condition 3: Does $\lim_{x\to 2} f(x) = f(2)$?

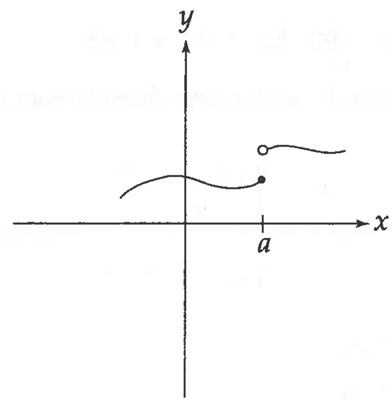
The $\lim_{x\to 2} f(x) = 3$, but $f(2) = 4$. Because these aren't equal, the answer is "no" and the function is not continuous at $x = 2$.

TYPES OF DISCONTINUITIES

There are four types of discontinuities you have to know: jump, point, essential, and removable.

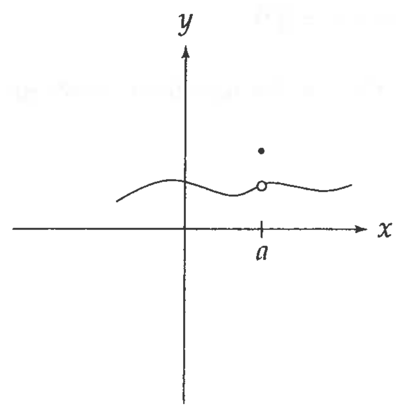
A **jump** discontinuity occurs when the curve "breaks" at a particular place and starts somewhere else. In other words, $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

A sample looks like this:



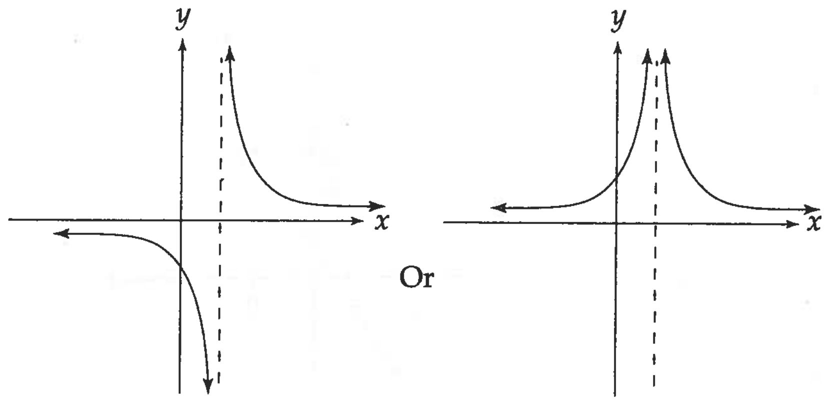
A **point** discontinuity occurs when the curve has a "hole" in it from a missing point because the function has a value at that point that's "off the curve." In other words, $\lim_{x \to a} f(x) \neq f(a)$.

Here's what a point discontinuity looks like:



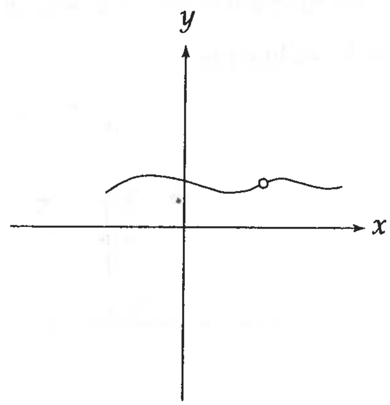
An **essential** discontinuity occurs when the curve has a vertical asymptote.

Like so:



A **removable** discontinuity occurs when you have a rational expression with common factors in the numerator and denominator. Because these factors can be canceled, the discontinuity is "removable."

Here's an example:



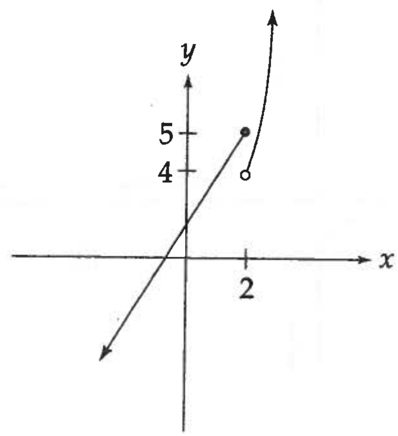
This curve looks very similar to a point discontinuity, but notice that with a removable discontinuity, $f(x)$ is not defined at the point, whereas with a point discontinuity, $f(x)$ is defined there.

Now that you know what these four types of discontinuities look like, let's see what types of functions are not everywhere continuous.

Example 5: Consider the function:

$$f(x) = \begin{cases} x+3, & x \leq 2 \\ x^2, & x > 2 \end{cases}$$

The left-hand limit is 5 as x approaches 2, and the right-hand limit is 4 as x approaches 2. Because the curve has different values on each side of 2, the curve is discontinuous at $x = 2$. We say that the curve "jumps" at $x = 2$ from the left-hand curve to the right-hand curve because the left and right-hand limits differ. It looks like this:

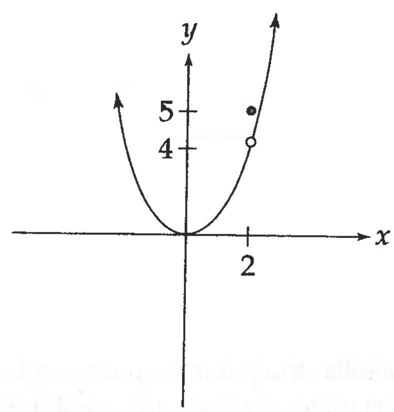


This is an example of a jump discontinuity.

Example 6: Consider the function:

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 5, & x = 2 \end{cases}$$

Because $\lim\limits_{x \to 2} f(x) \neq f(2)$; the function is discontinuous at $x = 2$. The curve is continuous everywhere except at the point $x = 2$. It looks like this:

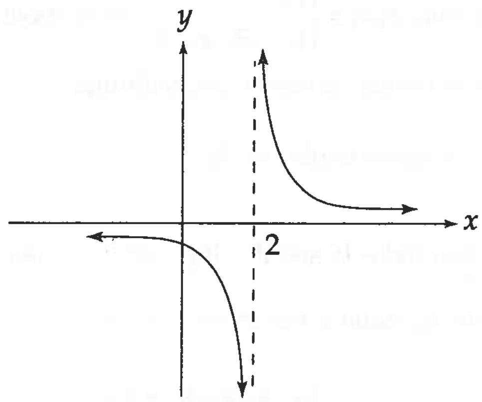


This is an example of a point discontinuity.

Example 7: Consider the function: $f(x) = \dfrac{5}{x-2}$

The function is discontinuous because it's possible for the denominator to equal zero (at $x = 2$). This means that $f(2)$ doesn't exist, and the function has an asymptote at $x = 2$. In addition, $\lim\limits_{x \to 2^-} f(x) = -\infty$ and $\lim\limits_{x \to 2^+} f(x) = \infty$

The graph looks like this:



This is an example of an essential discontinuity.

Example 8: Consider the function:

$$f(x) = \frac{x^2 - 8x + 15}{x^2 - 6x + 5}$$

If you factor the top and bottom, you can see where the discontinuities are:

$$f(x) = \frac{x^2 - 8x + 15}{x^2 - 6x + 5} = \frac{(x-3)(x-5)}{(x-1)(x-5)}$$

The function has a zero in the denominator when $x = 1$ or $x = 5$, so the function is discontinuous at those two points. But, you can cancel the term $(x - 5)$ from both the numerator and the denominator, leaving you with:

$$f(x) = \frac{x-3}{x-1}$$

Now the reduced function *is* continuous at $x = 5$. Thus the original function has a removable discontinuity at $x = 5$. Furthermore, if you now plug $x = 5$ into the reduced function, you get:

$$f(5) = \frac{2}{4} = \frac{1}{2}$$

The discontinuity is at $x = 5$, and there's a hole at $\left(5, \frac{1}{2}\right)$. In other words, if the original function were continuous at $x = 5$, it would have the value $\frac{1}{2}$. Notice that this is the same as: $\lim_{x \to 5} f(x)$.

These are the types of discontinuities that you can expect to encounter on the AP examination. Here are some sample problems and their solutions. Cover the answers as you work, then check your results.

Problem 1. Is the function $f(x)=\begin{cases}2x^3-1, & x<2\\ 6x-3, & x\geq 2\end{cases}$ continuous at $x = 2$?

Answer: Test the conditions necessary for continuity.

Condition 1: $f(2) = 9$, so we're okay so far.

Condition 2: The $\lim_{x\to 2^-} f(x) = 15$ and the $\lim_{x\to 2^+} f(x) = 9$. These two limits don't agree, so the $\lim_{x\to 2} f(x)$ doesn't exist and the function is not continuous at $x = 2$.

Problem 2. Is the function $f(x)=\begin{cases}x^2+3x+5, & x<1\\ 6x+3, & x\geq 1\end{cases}$ continuous at $x = 1$?

Answer: Condition 1: $f(1) = 9$.

Condition 2: The $\lim_{x\to 1^-} f(x)=9$ and the $\lim_{x\to 1^+} f(x)=9$.

Therefore, the $\lim_{x\to 1} f(x)$ exists and is equal to 9.

Condition 3: $\lim_{x\to 1} f(x) = f(1) = 9$.

The function satisfies all three conditions, so it is continuous at $x = 1$.

Problem 3. For what value of a is the function $f(x)=\begin{cases}ax+5, & x<4\\ x^2-x, & x\geq 4\end{cases}$ continuous at $x = 4$?

Answer: Because $f(4) = 12$, the function passes the first condition.

For Condition 2 to be satisfied, the $\lim_{x\to 4^-} f(x) = 4a+5$ must equal the $\lim_{x\to 4^+} f(x)=12$. So, set $4a + 5 = 12$. If $a = \frac{7}{4}$, the limit will exist at $x = 4$ and the other two conditions will also be fulfilled.

Therefore, the value $a = \frac{7}{4}$ makes the function continuous at $x = 4$.

Problem 4. Where does the function $f(x)=\dfrac{2x^2-7x-15}{x^2-x-20}$

have: (a) an essential discontinuity; and (b) a removable discontinuity?

Answer: If you factor the top and bottom of this fraction, you get:

$$f(x)=\frac{2x^2-7x-15}{x^2-x-20}=\frac{(2x+3)(x-5)}{(x+4)(x-5)}$$

Thus, the function has an essential discontinuity at $x = -4$. If we then cancel the term $(x - 5)$, and substitute $x = 5$ into the reduced expression, we get $f(5)=\frac{13}{9}$. Therefore, the function has a removable discontinuity at $\left(5,\frac{13}{9}\right)$.

Note: Don't confuse coordinate parentheses with interval notation. In interval notation, square brackets include endpoints and parentheses do not. For example, the interval $2 \le x \le 4$ is written $[2, 4]$ and the interval $2 < x < 4$ is written $(2, 4)$.

PRACTICE PROBLEM SET 2

Now try these problems. The answers are in Chapter 21.

1. Is the function $f(x)=\begin{cases} x+7, & x<2 \\ 9, & x=2 \\ 3x+3, & x>2 \end{cases}$ continuous at $x = 2$?

2. Is the function $f(x)=\begin{cases} 4x^2-2x, & x<3 \\ 10x-1, & x=3 \\ 30, & x>3 \end{cases}$ continuous at $x = 3$?

3. Is the function $f(x)=\begin{cases} 5x+7, & x<3 \\ 7x+1, & x>3 \end{cases}$ continuous at $x = 3$?

4. Is the function $f(x) = \sec x$ continuous everywhere?

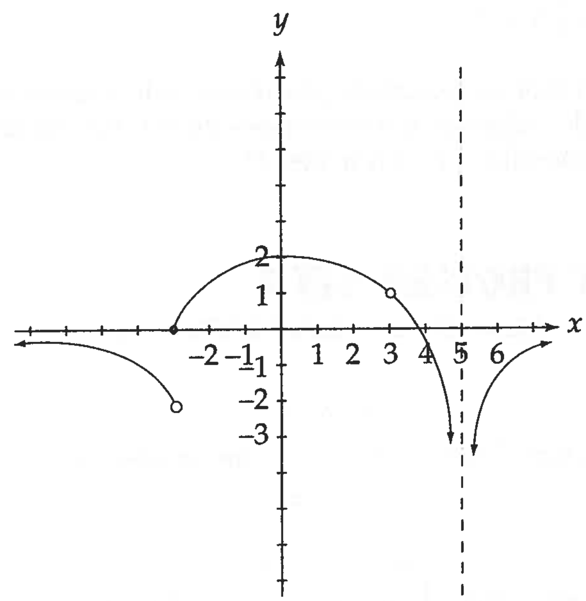
5. Is the function $f(x) = \sec x$ continuous on the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$?

6. Is the function $f(x) = \sec x$ continuous on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$?

7. For what value(s) of k is the function $f(x) = \begin{cases} 3x^2-11x-4, & x\le 4 \\ kx^2-2x-1, & x>4 \end{cases}$ continuous at $x = 4$?

8. For what value(s) of k is the function $f(x) = \begin{cases} -6x-12, & x<-3 \\ k^2-5k, & x=-3 \\ 6, & x>-3 \end{cases}$ continuous at $x = -3$?

9. At what point is the removable discontinuity for the function $f(x) = \dfrac{x^2+5x-24}{x^2-x-6}$?



10. Given the graph of $f(x)$ above, find:

(a) $\lim\limits_{x \to -\infty} f(x)$

(b) $\lim\limits_{x \to \infty} f(x)$

(c). $\lim\limits_{x \to 3^-} f(x)$

(d) $\lim\limits_{x \to 3^+} f(x)$

(e) $f(3)$

(f) Any discontinuities.